

# Algebraic points on Shimura curves of $\Gamma_0(p)$ -type (III)

Keisuke Arai

## Abstract

In previous articles, we classified the characters associated to algebraic points on Shimura curves of  $\Gamma_0(p)$ -type, and over number fields in a certain large class we showed that there are at most elliptic points on such a Shimura curve for every sufficiently large prime number  $p$ . In this article, we prove the non-existence of elliptic points on Shimura curves of  $\Gamma_0(p)$ -type under a mild assumption. We also give an explicit example.

## 1 Introduction

Let  $B$  be an indefinite quaternion division algebra over  $\mathbb{Q}$  of discriminant  $d$ . Fix a maximal order  $\mathcal{O}$  of  $B$ . A *QM-abelian surface by  $\mathcal{O}$*  over a scheme  $S$  is a pair  $(A, i)$  where  $A$  is an abelian scheme over  $S$  of relative dimension 2, and  $i : \mathcal{O} \hookrightarrow \text{End}_S(A)$  is an injective ring homomorphism (sending 1 to id) (cf. [6, p.591]). Here  $\text{End}_S(A)$  is the ring of endomorphisms of  $A$  defined over  $S$ . We assume that  $A$  has a left  $\mathcal{O}$ -action. We will sometimes omit “by  $\mathcal{O}$ ” and simply write “a QM-abelian surface” if there is no fear of confusion.

Let  $M^B$  be the coarse moduli scheme over  $\mathbb{Q}$  parameterizing isomorphism classes of QM-abelian surfaces by  $\mathcal{O}$  (cf. [7, p.93]). Then  $M^B$  is a proper smooth curve over  $\mathbb{Q}$ , called a *Shimura curve*. For a prime number  $p$  not dividing  $d$ , let  $M_0^B(p)$  be the coarse moduli scheme over  $\mathbb{Q}$  parameterizing isomorphism classes of triples  $(A, i, V)$  where  $(A, i)$  is a QM-abelian surface by  $\mathcal{O}$  and  $V$  is a left  $\mathcal{O}$ -submodule of  $A[p]$  with  $\mathbb{F}_p$ -dimension 2. Here  $A[p]$  is the kernel of multiplication by  $p$  in  $A$ . Then  $M_0^B(p)$  is a proper smooth curve over  $\mathbb{Q}$ , which we call a *Shimura curve of  $\Gamma_0(p)$ -type*. We have a natural map

$$\pi^B(p) : M_0^B(p) \longrightarrow M^B$$

over  $\mathbb{Q}$  defined by  $(A, i, V) \longmapsto (A, i)$ . Note that  $M^B$  (resp.  $M_0^B(p)$ ) is an analogue of the modular curve  $X_0(1)$  (resp.  $X_0(p)$ ).

We say that a prime of a number field is *of odd degree* if the cardinality of the residue field is an odd power of the residual characteristic. The main result of this article is:

**Theorem 1.1.** *Let  $k$  be a finite Galois extension of  $\mathbb{Q}$  which does not contain the Hilbert class field of any imaginary quadratic field. Assume that there is a prime  $\mathfrak{q}$*

of  $k$  such that  $\mathfrak{q}$  is of odd degree, the residual characteristic  $q$  of  $\mathfrak{q}$  is unramified in  $k$ , and  $B \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{-q}) \not\cong M_2(\mathbb{Q}(\sqrt{-q}))$ . Then  $M_0^B(p)(k) = \emptyset$  holds for every sufficiently large prime number  $p$ .

**Remark 1.2.** (1) Theorem 1.1 is an analogue of the results for points on the modular curve  $X_0(p)$  (cf. [9], [10]).

- (2) We see that the Riemann surface  $M_0^B(p)(\mathbb{C})$  is isomorphic to a quotient of the upper half-plane, and it often has elliptic points of order 2 or 3 (cf. [4, §3]).
- (3) In previous articles [2], [3], [4] (cf. [5]), we could not exclude the possibility of the existence of elliptic points in  $M_0^B(p)(k)$ .
- (4) We see  $M^B(\mathbb{R}) = \emptyset$  by [13, Theorem 0]. Since there is a map  $\pi^B(p) : M_0^B(p) \longrightarrow M^B$  over  $\mathbb{Q}$ , we have  $M_0^B(p)(\mathbb{R}) = \emptyset$  for any prime number  $p$  (not dividing  $d$ ).

### Notation

For a field  $F$ , let  $\text{char } F$  denote the characteristic of  $F$ , let  $\overline{F}$  denote an algebraic closure of  $F$ , let  $F^{\text{sep}}$  (resp.  $F^{\text{ab}}$ ) denote the separable closure (resp. the maximal abelian extension) of  $F$  inside  $\overline{F}$ , and let  $G_F = \text{Gal}(F^{\text{sep}}/F)$ ,  $G_F^{\text{ab}} = \text{Gal}(F^{\text{ab}}/F)$ . For a prime number  $p$  and a field  $F$  with  $\text{char } F \neq p$ , let  $\theta_p : G_F \longrightarrow \mathbb{F}_p^\times$  denote the mod  $p$  cyclotomic character.

For a number field  $k$ , let  $\mathcal{O}_k$  denote the ring of integers of  $k$ ; put  $N(\mathfrak{q}) := \sharp(\mathcal{O}_k/\mathfrak{q})$  for a prime  $\mathfrak{q}$  of  $k$ ; let  $Cl_k$  denote the ideal class group of  $k$ ; let  $h_k$  denote the class number of  $k$ ; fix an inclusion  $k \hookrightarrow \mathbb{C}$  and take the algebraic closure  $\overline{k}$  inside  $\mathbb{C}$ ; let  $k_v$  denote the completion of  $k$  at  $v$  where  $v$  is a place (or a prime) of  $k$ ; and let  $\text{Ram}(k)$  denote the set of prime numbers which are ramified in  $k$ .

## 2 Galois representations associated to QM-abelian surfaces (generalities)

We review [4, §3] briefly in order to consider the Galois representations associated to a QM-abelian surface. Take a prime number  $p$  not dividing  $d$ . Let  $F$  be a field with  $\text{char } F \neq p$ . Let  $(A, i)$  be a QM-abelian surface by  $\mathcal{O}$  over  $F$ . The action of  $G_F$  on  $A[p](F^{\text{sep}}) \cong \mathbb{F}_p^4$  determines a representation  $\overline{\rho} : G_F \longrightarrow \text{GL}_4(\mathbb{F}_p)$ . By a suitable choice of basis,  $\overline{\rho}$  factors as

$$\overline{\rho} : G_F \longrightarrow \left\{ \begin{pmatrix} sI_2 & tI_2 \\ uI_2 & vI_2 \end{pmatrix} \mid \begin{pmatrix} s & t \\ u & v \end{pmatrix} \in \text{GL}_2(\mathbb{F}_p) \right\} \subseteq \text{GL}_4(\mathbb{F}_p),$$

where  $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ . Let

$$\overline{\rho}_{A,p} : G_F \longrightarrow \text{GL}_2(\mathbb{F}_p) \tag{2.1}$$

denote the Galois representation induced from  $\overline{\rho}$  by “ $\begin{pmatrix} s & t \\ u & v \end{pmatrix}$ ”, so that we have

$$\overline{\rho}_{A,p}(\sigma) = \begin{pmatrix} s(\sigma) & t(\sigma) \\ u(\sigma) & v(\sigma) \end{pmatrix} \text{ if } \overline{\rho}(\sigma) = \begin{pmatrix} s(\sigma)I_2 & t(\sigma)I_2 \\ u(\sigma)I_2 & v(\sigma)I_2 \end{pmatrix} \text{ for } \sigma \in G_F.$$

Suppose that  $A[p](F^{\text{sep}})$  has a left  $\mathcal{O}$ -submodule  $V$  with  $\mathbb{F}_p$ -dimension 2 which is stable under the action of  $G_F$ . Then, by taking a conjugate if necessary, we may assume  $\bar{\rho}_{A,p}(G_F) \subseteq \left\{ \begin{pmatrix} s & t \\ 0 & v \end{pmatrix} \right\} \subseteq \text{GL}_2(\mathbb{F}_p)$ . Let

$$\lambda : G_F \longrightarrow \mathbb{F}_p^\times \quad (2.2)$$

denote the character induced from  $\bar{\rho}_{A,p}$  by “ $s$ ”, so that  $\bar{\rho}_{A,p}(\sigma) = \begin{pmatrix} \lambda(\sigma) & * \\ 0 & * \end{pmatrix}$  for  $\sigma \in G_F$ . Note that  $G_F$  acts on  $V$  by  $\lambda$  (i.e.  $\bar{\rho}(\sigma)(v) = \lambda(\sigma)v$  for  $\sigma \in G_F, v \in V$ ).

### 3 Automorphism groups

We give a brief summary of [4, §4] concerning the automorphism groups of a QM-abelian surface. Let  $(A, i)$  be a QM-abelian surface by  $\mathcal{O}$  over a field  $F$ . Let  $\text{End}(A)$  (resp.  $\text{Aut}(A)$ ) denote the ring of endomorphisms (the group of automorphisms) of  $A$  defined over  $\bar{F}$ . Put

$$\text{End}_{\mathcal{O}}(A) := \{f \in \text{End}(A) \mid f \circ i(g) = i(g) \circ f \text{ for any } g \in \mathcal{O}\},$$

$$\text{Aut}_{\mathcal{O}}(A) := \text{Aut}(A) \cap \text{End}_{\mathcal{O}}(A).$$

If  $\text{char } F = 0$ , then  $\text{Aut}_{\mathcal{O}}(A) \cong \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}$  or  $\mathbb{Z}/6\mathbb{Z}$ .

Let  $p$  be a prime number not dividing  $d$ . Let  $(A, i, V)$  be a triple where  $(A, i)$  is a QM-abelian surface by  $\mathcal{O}$  over a field  $F$  and  $V$  is a left  $\mathcal{O}$ -submodule of  $A[p](\bar{F})$  with  $\mathbb{F}_p$ -dimension 2. Define a subgroup  $\text{Aut}_{\mathcal{O}}(A, V)$  of  $\text{Aut}_{\mathcal{O}}(A)$  by

$$\text{Aut}_{\mathcal{O}}(A, V) := \{f \in \text{Aut}_{\mathcal{O}}(A) \mid f(V) = V\}.$$

Assume  $\text{char } F = 0$ . Then  $\text{Aut}_{\mathcal{O}}(A, V) \cong \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}$  or  $\mathbb{Z}/6\mathbb{Z}$ . Note that we have  $\text{Aut}_{\mathcal{O}}(A) \cong \mathbb{Z}/2\mathbb{Z}$  (resp.  $\text{Aut}_{\mathcal{O}}(A, V) \cong \mathbb{Z}/2\mathbb{Z}$ ) if and only if  $\text{Aut}_{\mathcal{O}}(A) = \{\pm 1\}$  (resp.  $\text{Aut}_{\mathcal{O}}(A, V) = \{\pm 1\}$ ).

### 4 Fields of definition

We review [4, §4] to consider the field of definition of a point on  $M_0^B(p)$ . Let  $k$  be a number field. Let  $p$  be a prime number not dividing  $d$ . Take a point

$$x \in M_0^B(p)(k).$$

Let  $x' \in M^B(k)$  be the image of  $x$  by the map  $\pi^B(p) : M_0^B(p) \longrightarrow M^B$ . Then  $x'$  is represented by a QM-abelian surface (say  $(A_x, i_x)$ ) over  $\bar{k}$ , and  $x$  is represented by a triple  $(A_x, i_x, V_x)$  where  $V_x$  is a left  $\mathcal{O}$ -submodule of  $A[p](\bar{k})$  with  $\mathbb{F}_p$ -dimension 2. For a finite extension  $M$  of  $k$ , we say that *we can take  $(A_x, i_x, V_x)$  to be defined over  $M$*  if there is a QM-abelian surface  $(A, i)$  over  $M$  and a left  $\mathcal{O}$ -submodule  $V$  of  $A[p](\bar{k})$  with  $\mathbb{F}_p$ -dimension 2 stable under the action of  $G_M$  such that there is an

isomorphism between  $(A, i) \otimes_M \bar{k}$  and  $(A_x, i_x)$  under which  $V$  corresponds to  $V_x$ . Put

$$\text{Aut}(x) := \text{Aut}_{\mathcal{O}}(A_x, V_x), \quad \text{Aut}(x') := \text{Aut}_{\mathcal{O}}(A_x).$$

Then  $\text{Aut}(x)$  is a subgroup of  $\text{Aut}(x')$ . Note that  $x$  is an elliptic point of order 2 (resp. 3) if and only if  $\text{Aut}(x) \cong \mathbb{Z}/4\mathbb{Z}$  (resp.  $\text{Aut}(x) \cong \mathbb{Z}/6\mathbb{Z}$ ). Since  $x$  is a  $k$ -rational point, we have  ${}^\sigma x = x$  for any  $\sigma \in G_k$ . Then, for any  $\sigma \in G_k$ , there is an isomorphism

$$\phi_\sigma : {}^\sigma(A_x, i_x, V_x) \longrightarrow (A_x, i_x, V_x),$$

which we fix once for all. For  $\sigma, \tau \in G_k$ , put

$$c_x(\sigma, \tau) := \phi_\sigma \circ {}^\sigma \phi_\tau \circ \phi_{\sigma\tau}^{-1} \in \text{Aut}(x).$$

Then  $c_x$  is a 2-cocycle and defines a cohomology class  $[c_x] \in H^2(G_k, \text{Aut}(x))$ . Here the action of  $G_k$  on  $\text{Aut}(x)$  is defined in a natural manner (cf. [4, §4]). For a place  $v$  of  $k$ , let  $[c_x]_v \in H^2(G_{k_v}, \text{Aut}(x))$  denote the restriction of  $[c_x]$  to  $G_{k_v}$ .

**Proposition 4.1** ([4, Proposition 4.2]). (1) *Suppose  $B \otimes_{\mathbb{Q}} k \cong M_2(k)$ . Further assume  $\text{Aut}(x) \neq \{\pm 1\}$  or  $\text{Aut}(x') \not\cong \mathbb{Z}/4\mathbb{Z}$ . Then we can take  $(A_x, i_x, V_x)$  to be defined over  $k$ .*

(2) *Assume  $\text{Aut}(x) = \{\pm 1\}$ . Then there is a quadratic extension  $K$  of  $k$  such that we can take  $(A_x, i_x, V_x)$  to be defined over  $K$ .*

**Lemma 4.2** ([4, Lemma 4.3]). *Let  $K$  be a quadratic extension of  $k$ . Assume  $\text{Aut}(x) = \{\pm 1\}$ . Then the following two conditions are equivalent:*

- (1) *We can take  $(A_x, i_x, V_x)$  to be defined over  $K$ .*
- (2) *For any place  $v$  of  $k$  satisfying  $[c_x]_v \neq 0$ , the tensor product  $K \otimes_k k_v$  is a field.*

## 5 Classification of characters

We keep the notation in Section 4. Throughout this section, assume  $\text{Aut}(x) = \{\pm 1\}$ . Let  $K$  be a quadratic extension of  $k$  which satisfies the equivalent conditions in Lemma 4.2. Then  $x$  is represented by a triple  $(A, i, V)$ , where  $(A, i)$  is a QM-abelian surface over  $K$  and  $V$  is a left  $\mathcal{O}$ -submodule of  $A[p](\bar{k})$  with  $\mathbb{F}_p$ -dimension 2 stable under the action of  $G_K$ . Let

$$\lambda : G_K \longrightarrow \mathbb{F}_p^\times$$

be the character associated to  $V$  in (2.2). Let  $\lambda^{\text{ab}} : G_K^{\text{ab}} \longrightarrow \mathbb{F}_p^\times$  be the natural map induced from  $\lambda$ . Put

$$\varphi := \lambda^{\text{ab}} \circ \text{tr}_{K/k} : G_k \longrightarrow G_K^{\text{ab}} \longrightarrow \mathbb{F}_p^\times, \quad (5.1)$$

where  $\text{tr}_{K/k} : G_k \longrightarrow G_K^{\text{ab}}$  is the transfer map. By [4, Lemma 5.1] (resp. [4, Corollary 5.2]), the character  $\lambda^{12}$  (resp.  $\varphi^{12}$ ) is unramified at every prime of  $K$  (resp.  $k$ ) not dividing  $p$ , and so it corresponds to a character of the ideal group  $\mathfrak{I}_K(p)$  (resp.  $\mathfrak{I}_k(p)$ ) consisting of fractional ideals of  $K$  (resp.  $k$ ) prime to  $p$ . By abuse of notation, let  $\lambda^{12}$  (resp.  $\varphi^{12}$ ) also denote the corresponding character of  $\mathfrak{I}_K(p)$  (resp.  $\mathfrak{I}_k(p)$ ).

Let  $\mathcal{M}^{new}$  be the set of prime numbers which split completely in  $k$ . Let  $\mathcal{N}^{new}$  be the set of primes of  $k$  which divide some prime number  $q \in \mathcal{M}^{new}$ . Take a finite subset  $\emptyset \neq \mathcal{S}^{new} \subseteq \mathcal{N}^{new}$  which generates  $Cl_k$ . For each prime  $\mathfrak{q} \in \mathcal{S}^{new}$ , fix an element  $\alpha_{\mathfrak{q}} \in \mathcal{O}_k \setminus \{0\}$  satisfying  $\mathfrak{q}^{h_k} = \alpha_{\mathfrak{q}} \mathcal{O}_k$ . For an integer  $n \geq 1$ , put

$$\mathcal{FR}(n) := \{ \beta \in \mathbb{C} \mid \beta^2 + a\beta + n = 0 \text{ for some integer } a \in \mathbb{Z} \text{ with } |a| \leq 2\sqrt{n} \}.$$

For any element  $\beta \in \mathcal{FR}(n)$ , we have  $|\beta| = \sqrt{n}$ . When  $k$  is Galois over  $\mathbb{Q}$ , define the sets

$$\begin{aligned} \mathcal{E}(k) &:= \left\{ \varepsilon_0 = \sum_{\sigma \in \text{Gal}(k/\mathbb{Q})} a_{\sigma} \sigma \in \mathbb{Z}[\text{Gal}(k/\mathbb{Q})] \mid a_{\sigma} \in \{0, 8, 12, 16, 24\} \right\}, \\ \mathcal{M}_1^{new}(k) &:= \{ (\mathfrak{q}, \varepsilon_0, \beta_{\mathfrak{q}}) \mid \mathfrak{q} \in \mathcal{S}^{new}, \varepsilon_0 \in \mathcal{E}(k), \beta_{\mathfrak{q}} \in \mathcal{FR}(N(\mathfrak{q})) \}, \\ \mathcal{M}_2^{new}(k) &:= \{ \text{Norm}_{k(\beta_{\mathfrak{q}})/\mathbb{Q}}(\alpha_{\mathfrak{q}}^{\varepsilon_0} - \beta_{\mathfrak{q}}^{24h_k}) \in \mathbb{Z} \mid (\mathfrak{q}, \varepsilon_0, \beta_{\mathfrak{q}}) \in \mathcal{M}_1^{new}(k) \} \setminus \{0\}, \\ \mathcal{N}_0^{new}(k) &:= \{ l : \text{prime number} \mid l \text{ divides some integer } m \in \mathcal{M}_2^{new}(k) \}, \\ \mathcal{T}^{new}(k) &:= \{ l' : \text{prime number} \mid l' \text{ is divisible by some prime } \mathfrak{q}' \in \mathcal{S}^{new} \} \cup \{2, 3\}, \\ \mathcal{N}_1^{new}(k) &:= \mathcal{N}_0^{new}(k) \cup \mathcal{T}^{new}(k) \cup \text{Ram}(k). \end{aligned}$$

Note that all the sets,  $\mathcal{FR}(n)$ ,  $\mathcal{E}(k)$ ,  $\mathcal{M}_1^{new}(k)$ ,  $\mathcal{M}_2^{new}(k)$ ,  $\mathcal{N}_0^{new}(k)$ ,  $\mathcal{T}^{new}(k)$ , and  $\mathcal{N}_1^{new}(k)$ , are finite. In [3], an upper bound of  $\mathcal{N}_1^{new}(k)$  is given. We have the following classification of  $\varphi$ :

**Theorem 5.1** ([3, Theorem 5.1]). *Assume that  $k$  is Galois over  $\mathbb{Q}$ . If  $p \notin \mathcal{N}_1^{new}(k)$  (and if  $p$  does not divide  $d$ ), then the character  $\varphi : G_k \longrightarrow \mathbb{F}_p^{\times}$  is of one of the following types:*

Type 2.  $\varphi^{12} = \theta_p^{12}$  and  $p \equiv 3 \pmod{4}$ .

Type 3. *There is an imaginary quadratic field  $L$  satisfying the following conditions:*

- (a) *The Hilbert class field  $H_L$  of  $L$  is contained in  $k$ .*
- (b) *There is a prime  $\mathfrak{p}_L$  of  $L$  lying over  $p$  such that  $\varphi^{12}(\mathfrak{a}) \equiv \delta^{24} \pmod{\mathfrak{p}_L}$  holds for any fractional ideal  $\mathfrak{a}$  of  $k$  prime to  $p$ . Here  $\delta$  is any element of  $L$  such that  $\text{Norm}_{k/L}(\mathfrak{a}) = \delta \mathcal{O}_L$ .*

**Lemma 5.2** ([3, Lemma 5.2]). *Suppose that  $k$  is Galois over  $\mathbb{Q}$ , and  $p \geq 11$ ,  $p \neq 13$ ,  $p \notin \mathcal{N}_1^{new}(k)$ . Further assume the following two conditions:*

- (a) *Every prime of  $k$  above  $p$  is inert in  $K/k$ .*
- (b) *Every prime of  $k$  in  $\mathcal{S}^{new}$  is ramified in  $K/k$ .*

*If  $\varphi$  is of type 2, then we have the following assertions:*

- (i) *The character  $\lambda^{12} \theta_p^{-6} : G_K \longrightarrow \mathbb{F}_p^{\times}$  is unramified everywhere.*
- (ii) *The map  $Cl_K \longrightarrow \mathbb{F}_p^{\times}$  induced from  $\lambda^{12} \theta_p^{-6}$  is trivial on  $C_{K/k} := \text{Im}(Cl_k \longrightarrow Cl_K)$ , where  $Cl_k \longrightarrow Cl_K$  is the map defined by  $[\mathfrak{a}] \longmapsto [\mathfrak{a} \mathcal{O}_K]$ .*

From now to the end of this section, suppose that  $k$  is Galois over  $\mathbb{Q}$ ,  $p \geq 11$ ,  $p \neq 13$ ,  $p \notin \mathcal{N}_1^{new}(k)$ , and that  $\varphi$  is of type 2. Let  $q \neq p$  be a prime number, and take a prime  $\mathfrak{q}$  of  $k$  above  $q$ . By replacing  $K$  if necessary, we may assume the conditions (a), (b) in Lemma 5.2 and that  $\mathfrak{q}$  is ramified in  $K/k$  (cf. [4, Remark 4.4]).

Let  $\mathfrak{q}_K$  be the unique prime of  $K$  above  $\mathfrak{q}$ . The abelian surface  $A \otimes_K K_{\mathfrak{q}_K}$  has good reduction after a totally ramified finite extension  $M/K_{\mathfrak{q}_K}$  (cf. [7, Proposition 3.2]). Let  $\tilde{A}$  be the special fiber of the Néron model of  $A \otimes_K M$ . Then  $\tilde{A}$  is a QM-abelian surface by  $\mathcal{O}$  over  $\mathcal{O}_k/\mathfrak{q}$ . We have  $\lambda(\text{Frob}_M) \equiv \beta$  modulo a prime  $\mathfrak{p}_0$  of  $\mathbb{Q}(\beta)$  above  $p$  for a Frobenius eigenvalue  $\beta$  of  $\tilde{A}$ , where  $\text{Frob}_M$  is the arithmetic Frobenius of  $G_M (\subseteq G_{K_{\mathfrak{q}_K}})$ . We see  $\beta \in \mathcal{FR}(\mathbb{N}(\mathfrak{q}))$  by [7, p.97]. Since  $\det \bar{\rho}_{A,p} = \theta_p$  (cf. [11, Proposition 1.1 (2)]), we have  $\lambda^{-1}\theta_p(\text{Frob}_M) \equiv \bar{\beta} \pmod{\mathfrak{p}_0}$ , where  $\bar{\beta}$  is the complex conjugate of  $\beta$ . Put

$$\psi := \lambda\theta_p^{-\frac{p+1}{4}}.$$

Then  $\psi^{12} = \lambda^{12}\theta_p^{-3(p+1)} = \lambda^{12}\theta_p^{-6}$ .

- Lemma 5.3.** (1)  $\psi(\text{Frob}_M)^6 = 1$ .  
(2)  $\psi(\text{Frob}_M)^2 + \psi(\text{Frob}_M)^{-2} = -1$  or  $2$ .  
(3)  $\beta^2 + \bar{\beta}^2 \equiv -\mathbb{N}(\mathfrak{q})^{\frac{p+1}{2}}$  or  $2\mathbb{N}(\mathfrak{q})^{\frac{p+1}{2}} \pmod{p}$ .

*Proof.* (1) By Lemma 5.2 (ii), we have  $1 = \lambda^{12}(\mathfrak{q}\mathcal{O}_K)\theta_p^{-6}(\mathfrak{q}\mathcal{O}_K) = \psi^{12}(\mathfrak{q}\mathcal{O}_K) = \psi^{24}(\mathfrak{q}_K) = \psi^{24}(\text{Frob}_M) = \psi(\text{Frob}_M)^{24}$ . Note that the fourth equality holds because the extension  $M/K_{\mathfrak{q}_K}$  is totally ramified. Since  $\mathbb{F}_p^\times$  is a cyclic group of order  $p-1$  and  $p-1 \equiv 2 \pmod{4}$ , we obtain  $\psi(\text{Frob}_M)^6 = 1$ .

(2) This follows immediately from (1).

(3)  $\beta^2 + \bar{\beta}^2 \equiv \psi(\text{Frob}_M)^2\theta_p(\text{Frob}_M)^{\frac{p+1}{2}} + \psi(\text{Frob}_M)^{-2}\theta_p(\text{Frob}_M)^{-\frac{p+3}{2}}$   
 $= \theta_p(\text{Frob}_M)^{\frac{p+1}{2}}(\psi(\text{Frob}_M)^2 + \psi(\text{Frob}_M)^{-2}) = -\mathbb{N}(\mathfrak{q})^{\frac{p+1}{2}}$  or  $2\mathbb{N}(\mathfrak{q})^{\frac{p+1}{2}} \pmod{p}$ . □

We repeat the argument in [2, §5] when  $\mathfrak{q}$  is of odd degree as follows:

**Lemma 5.4.** Suppose that  $\mathfrak{q}$  is of odd degree and satisfies  $\mathbb{N}(\mathfrak{q}) < \frac{p}{4}$ . Then:

- (1)  $\mathbb{N}(\mathfrak{q})^{\frac{p-1}{2}} \equiv -1 \pmod{p}$ .  
(2)  $(\beta + \bar{\beta})^2 \equiv 3\mathbb{N}(\mathfrak{q})$  or  $0 \pmod{p}$ .  
(3)  $q = 3$  and  $|\beta + \bar{\beta}| = \sqrt{3\mathbb{N}(\mathfrak{q})}$ , or  $\beta + \bar{\beta} = 0$ .  
(4)  $B \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{-q}) \cong M_2(\mathbb{Q}(\sqrt{-q}))$ .

*Proof.* (1) Assume otherwise i.e.  $\mathbb{N}(\mathfrak{q})^{\frac{p-1}{2}} \equiv 1 \pmod{p}$ . Then Lemma 5.3 (3) implies  $\beta^2 + \bar{\beta}^2 \equiv -\mathbb{N}(\mathfrak{q})$  or  $2\mathbb{N}(\mathfrak{q}) \pmod{p}$ , and so  $(\beta + \bar{\beta})^2 \equiv \mathbb{N}(\mathfrak{q})$  or  $4\mathbb{N}(\mathfrak{q}) \pmod{p}$ . Since  $\beta \in \mathcal{FR}(\mathbb{N}(\mathfrak{q}))$ , we have  $|\beta + \bar{\beta}| \leq 2\sqrt{\mathbb{N}(\mathfrak{q})}$ . Then  $|(\beta + \bar{\beta})^2 - \mathbb{N}(\mathfrak{q})| \leq 3\mathbb{N}(\mathfrak{q}) < p$  and  $|(\beta + \bar{\beta})^2 - 4\mathbb{N}(\mathfrak{q})| \leq 4\mathbb{N}(\mathfrak{q}) < p$ . Hence  $(\beta + \bar{\beta})^2 = \mathbb{N}(\mathfrak{q})$  or  $4\mathbb{N}(\mathfrak{q})$ . Since  $\mathfrak{q}$  is of odd degree, this contradicts  $\beta + \bar{\beta} \in \mathbb{Z}$ . Therefore we conclude  $\mathbb{N}(\mathfrak{q})^{\frac{p-1}{2}} \equiv -1 \pmod{p}$ .

(2) By (1) and Lemma 5.3 (3), we have  $\beta^2 + \bar{\beta}^2 \equiv \mathbb{N}(\mathfrak{q})$  or  $-2\mathbb{N}(\mathfrak{q}) \pmod{p}$ . Therefore  $(\beta + \bar{\beta})^2 \equiv 3\mathbb{N}(\mathfrak{q})$  or  $0 \pmod{p}$ .

(3) We have  $(\beta + \bar{\beta})^2 \leq 4\mathbb{N}(\mathfrak{q})$ . First assume  $(\beta + \bar{\beta})^2 \equiv 3\mathbb{N}(\mathfrak{q}) \pmod{p}$ . Then, since  $|(\beta + \bar{\beta})^2 - 3\mathbb{N}(\mathfrak{q})| \leq 3\mathbb{N}(\mathfrak{q}) < p$ , we have  $(\beta + \bar{\beta})^2 = 3\mathbb{N}(\mathfrak{q})$ . Therefore  $q = 3$  and  $|\beta + \bar{\beta}| = \sqrt{3\mathbb{N}(\mathfrak{q})}$ . Next assume  $(\beta + \bar{\beta})^2 \equiv 0 \pmod{p}$ . Then, since  $|(\beta + \bar{\beta})^2| \leq 4\mathbb{N}(\mathfrak{q}) < p$ , we have  $(\beta + \bar{\beta})^2 = 0$ . Therefore  $\beta + \bar{\beta} = 0$ .

(4) The number  $\beta$  is a Frobenius eigenvalue of the QM-abelian surface  $\tilde{A}$  by  $\mathcal{O}$  over  $\mathcal{O}_k/\mathfrak{q}$ , where  $\mathfrak{q}$  is of odd degree. Then, by (3) and [7, Theorem 2.1 (2) (4) and

Proposition 2.3], we conclude  $\text{End}_{\mathcal{O}_k/\mathfrak{q}}(\tilde{A}) \otimes_{\mathbb{Z}} \mathbb{Q} \cong M_2(\mathbb{Q}(\sqrt{-q})) \cong B \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{-q})$ . Here  $\text{End}_{\mathcal{O}_k/\mathfrak{q}}(\tilde{A})$  is the ring of endomorphisms of  $\tilde{A}$  defined over  $\mathcal{O}_k/\mathfrak{q}$ .  $\square$

## 6 Irreducibility of $\bar{\rho}_{A,p}$ and algebraic points on $M_0^B(p)$

Let  $k$  be a number field, and let  $(A, i)$  be a QM-abelian surface by  $\mathcal{O}$  over  $k$ . For a prime number  $p$  not dividing  $d$ , assume that the representation  $\bar{\rho}_{A,p}$  in (2.1) is reducible. Then there is a 1-dimensional sub-representation of  $\bar{\rho}_{A,p}$ , and let

$$\nu : G_k \longrightarrow \mathbb{F}_p^\times$$

be its associated character. In this case there is a left  $\mathcal{O}$ -submodule  $V$  of  $A[p](\bar{k})$  with  $\mathbb{F}_p$ -dimension 2 on which  $G_k$  acts by  $\nu$ , and so the triple  $(A, i, V)$  determines a point  $x \in M_0^B(p)(k)$ . Take any quadratic extension  $K$  of  $k$ . Then we have the characters  $\lambda : G_K \longrightarrow \mathbb{F}_p^\times$  and  $\varphi : G_k \longrightarrow \mathbb{F}_p^\times$  associated to the triple  $(A \otimes_k K, i, V)$ . Note that we have  $\varphi = \nu^2$  by construction of  $\varphi$ . The following theorems generalize [3, Theorems 6.1 and 6.2] slightly:

**Theorem 6.1.** *Let  $k$  be a finite Galois extension of  $\mathbb{Q}$  which does not contain the Hilbert class field of any imaginary quadratic field. Assume that there is a prime  $\mathfrak{q}$  of  $k$  such that  $\mathfrak{q}$  is of odd degree and the residual characteristic  $q$  of  $\mathfrak{q}$  satisfies  $B \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{-q}) \not\cong M_2(\mathbb{Q}(\sqrt{-q}))$ . Let  $p > 4N(\mathfrak{q})$  be a prime number which also satisfies  $p \geq 11$ ,  $p \neq 13$ ,  $p \nmid d$  and  $p \notin \mathcal{N}_1^{\text{new}}(k)$ . Then the representation*

$$\bar{\rho}_{A,p} : G_k \longrightarrow \text{GL}_2(\mathbb{F}_p)$$

*is irreducible.*

*Proof.* Assume that  $\bar{\rho}_{A,p}$  is reducible. Then the associated character  $\varphi$  is of type 2 in Theorem 5.1, because  $k$  does not contain the Hilbert class field of any imaginary quadratic field. By Lemma 5.4 (4) we have  $B \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{-q}) \cong M_2(\mathbb{Q}(\sqrt{-q}))$ , which is a contradiction.  $\square$

**Theorem 6.2.** *Let  $k$  be a finite Galois extension of  $\mathbb{Q}$  which does not contain the Hilbert class field of any imaginary quadratic field. Assume that there is a prime  $\mathfrak{q}$  of  $k$  such that  $\mathfrak{q}$  is of odd degree and the residual characteristic  $q$  of  $\mathfrak{q}$  satisfies  $B \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{-q}) \not\cong M_2(\mathbb{Q}(\sqrt{-q}))$ . Let  $p > 4N(\mathfrak{q})$  be a prime number which also satisfies  $p \geq 11$ ,  $p \neq 13$ ,  $p \nmid d$  and  $p \notin \mathcal{N}_1^{\text{new}}(k)$ .*

(1) *Suppose  $B \otimes_{\mathbb{Q}} k \cong M_2(k)$ . Then  $M_0^B(p)(k) = \emptyset$ .*

(2) *Suppose  $B \otimes_{\mathbb{Q}} k \not\cong M_2(k)$ . Then  $M_0^B(p)(k) \subseteq \{\text{elliptic points of order 2 or 3}\}$ .*

*Proof.* Take a point  $x \in M_0^B(p)(k)$ .

(1) (1-i) Assume  $\text{Aut}(x) \neq \{\pm 1\}$  or  $\text{Aut}(x') \not\cong \mathbb{Z}/4\mathbb{Z}$ . Then  $x$  is represented by a triple  $(A, i, V)$  defined over  $k$  by Proposition 4.1 (1), and the representation  $\bar{\rho}_{A,p}$  is reducible. This contradicts Theorem 6.1.

(1-ii) Assume otherwise (i.e.  $\text{Aut}(x) = \{\pm 1\}$  and  $\text{Aut}(x') \cong \mathbb{Z}/4\mathbb{Z}$ ). Then  $x$  is represented by a triple  $(A, i, V)$  defined over a quadratic extension of  $k$  by Proposition 4.1 (2), and we have a character  $\varphi : G_k \rightarrow \mathbb{F}_p^\times$  as in (5.1). By Theorem 5.1 and Lemma 5.4 (4), we have  $B \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{-q}) \cong M_2(\mathbb{Q}(\sqrt{-q}))$ . This is a contradiction.

(2) Assume that  $x$  is not an elliptic point of order 2 or 3. Then  $\text{Aut}(x) = \{\pm 1\}$ . By the same argument as in (1-ii), we have a contradiction.  $\square$

## 7 Elimination of elliptic points

In this section, we deduce Theorem 1.1 from Theorem 6.2.

**Proposition 7.1.** *Let  $k$  be a finite Galois extension of  $\mathbb{Q}$  which does not contain the Hilbert class field of any imaginary quadratic field. Assume that there is a prime  $\mathfrak{q}$  of  $k$  such that  $\mathfrak{q}$  is of odd degree, the residual characteristic  $q$  of  $\mathfrak{q}$  is unramified in  $k$ , and  $B \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{-q}) \not\cong M_2(\mathbb{Q}(\sqrt{-q}))$ . Then there is a finite Galois extension  $W$  of  $\mathbb{Q}$  satisfying the following conditions:*

- (i) *The composite field  $kW$  does not contain the Hilbert class field of any imaginary quadratic field.*
- (ii) *There is a prime  $\mathfrak{q}'$  of  $kW$  of odd degree with residual characteristic  $q$ .*
- (iii)  *$B \otimes_{\mathbb{Q}} kW \cong M_2(kW)$ .*

**Theorem 7.2.** *In the situation of Proposition 7.1, let  $p$  be a prime number satisfying  $p > 4N(\mathfrak{q}')$ ,  $p \geq 11$ ,  $p \neq 13$ ,  $p \nmid d$ , and  $p \notin \mathcal{N}_1^{\text{new}}(kW)$ . Then  $M_0^B(p)(k) = M_0^B(p)(kW) = \emptyset$ .*

*Proof.* Applying Theorem 6.2 (1) to  $kW$ , we obtain the result.  $\square$

Theorem 1.1 follows immediately from Theorem 7.2. From now to the end of this section, suppose that the assumption in Proposition 7.1 holds. Fix a prime  $\mathfrak{q}$  of  $k$  as in Proposition 7.1. Let  $\mathcal{U}$  be the set of non-zero integers  $N \in \mathbb{Z}$  such that

- $N$  is square free,
- $d \mid N$ ,
- $q \mid N$ .

For an integer  $N \in \mathbb{Z}$ , put  $W_N := \mathbb{Q}(\sqrt{N})$ .

**Lemma 7.3.** *Let  $N \in \mathcal{U}$ . Then:*

- (1)  *$B \otimes_{\mathbb{Q}} W_N \cong M_2(W_N)$ .*
- (2) *The prime  $\mathfrak{q}$  is ramified in  $kW_N$ .*
- (3) *If we let  $\mathfrak{q}'$  be the unique prime of  $kW_N$  above  $\mathfrak{q}$ , then  $\mathfrak{q}'$  is of odd degree.*



- Proof.* (1) This isomorphism holds because any prime divisor of  $d$  is ramified in  $W_N$ .  
(2) The prime number  $q$  is ramified in  $W_N$  and is *unramified* in  $k$ , as required.  
(3) By (2) we have  $N(\mathfrak{q}') = N(\mathfrak{q})$ , which is an odd power of  $q$ .

□

Then Proposition 7.1 is a consequence of the following lemma:

**Lemma 7.4.** *There is an integer  $N \in \mathcal{U}$  such that  $kW_N$  does not contain the Hilbert class field of any imaginary quadratic field.*

*Proof.* Assume otherwise i.e. for any  $N \in \mathcal{U}$ , assume that there is an imaginary quadratic field  $J_N$  such that  $kW_N$  contains the Hilbert class field  $H_N$  of  $J_N$ . Since  $H_N \not\subseteq k$ , we have  $k \subsetneq kH_N \subseteq kW_N$ . Then  $kH_N = kW_N$  because  $[kW_N : k] = 2$ . Then  $h_{J_N} = [H_N : J_N] = \frac{1}{2}[H_N : \mathbb{Q}] \leq \frac{1}{2}[kH_N : \mathbb{Q}] = \frac{1}{2}[kW_N : \mathbb{Q}] = [k : \mathbb{Q}]$ . We see that there are only finitely many such imaginary quadratic fields  $J_N$ . We also have  $kH_N = kW_N \supseteq W_N$ . Since  $\sharp \mathcal{U} = \infty$ , this implies that finitely many number fields contain infinitely many subfields, which is a contradiction.

□

## 8 Example

We give an example of Theorem 1.1 (or Theorem 7.2) as follows:

**Proposition 8.1.** *Suppose  $k = \mathbb{Q}(\zeta_{31})$  and  $d \in \{6, 22\}$ . Then:*

- (1)  $B \otimes_{\mathbb{Q}} k \not\cong M_2(k)$ .
- (2)  $\sharp M^B(k) = \infty$ .
- (3)  $kW_{-d}$  does not contain the Hilbert class field of any imaginary quadratic field.
- (4) If  $p > 128$  and  $p \notin \mathcal{N}_1^{new}(kW_{-d})$ , then  $M_0^B(p)(k) = M_0^B(p)(kW_{-d}) = \emptyset$ .

*Proof.* For a field  $F$  with  $\text{char } F \neq 2$  and two elements  $a, b \in F$ , let  $\left(\frac{a, b}{F}\right)$  be the quaternion algebra over  $F$  defined by

$$\left(\frac{a, b}{F}\right) = F + Fe + Ff + Fef, \quad e^2 = a, \quad f^2 = b, \quad ef = -fe.$$

For a prime number  $p$ , let  $e_p$  (resp.  $f_p$ , resp.  $g_p$ ) be the ramification index of  $p$  in  $k/\mathbb{Q}$  (resp. the degree of the residue field extension above  $p$  in  $k/\mathbb{Q}$ , resp. the number of primes of  $k$  above  $p$ ).

- (1) We have  $(e_2, f_2, g_2) = (1, 5, 6)$ . Let  $v$  be a place of  $k$  above 2.

[Case  $d = 6$ ]. We see  $B \cong \left(\frac{6, 5}{\mathbb{Q}}\right)$  by [12, §3.6 g)]. It suffices to prove  $B \otimes_{\mathbb{Q}} k_v \not\cong$

$M_2(k_v)$ . Since  $\mathbb{Q}_2(\sqrt{5})$  is the unramified quadratic extension of  $\mathbb{Q}_2$ , the prime number 5 is not a square in  $k_v$  (which is the unramified extension of  $\mathbb{Q}_2$  of degree 5). We also see that  $k_v(\sqrt{5})$  is the unramified quadratic extension of  $k_v$ . The 2-adic valuation

$$\nu : k_v^\times \longrightarrow \mathbb{Z} ; t \longmapsto \text{ord}_2(t)$$

induces an isomorphism  $\bar{\nu} : k_v^\times / \text{Norm}_{k_v(\sqrt{5})/k_v}(k_v(\sqrt{5})^\times) \cong \mathbb{Z}/2\mathbb{Z}$ . We have  $6 \notin \text{Norm}_{k_v(\sqrt{5})/k_v}(k_v(\sqrt{5})^\times)$  because  $\bar{\nu}(6) = 1 \neq 0$ . Therefore  $B \otimes_{\mathbb{Q}} k_v \not\cong M_2(k_v)$ , as required.

[Case  $d = 22$ ]. We have  $B \cong \left(\frac{22, 13}{\mathbb{Q}}\right)$  (loc.cit.). It suffices to prove  $B \otimes_{\mathbb{Q}} k_v \not\cong M_2(k_v)$ . By the same argument as in [Case  $d = 6$ ], the prime number 13 is not a square in  $k_v$ , and  $22 \notin \text{Norm}_{k_v(\sqrt{13})/k_v}(k_v(\sqrt{13})^\times)$ .

(2) The curve  $M^B$  is defined by  $x^2 + y^2 + 3z^2 = 0$  (resp.  $x^2 + y^2 + 11z^2 = 0$ ) in homogeneous coordinates if  $d = 6$  (resp.  $d = 22$ ) (cf. [8, Theorem 1-1]). For a prime number  $p$ , we have  $M^B(\mathbb{Q}_p) \neq \emptyset$  if and only if  $p \neq 3$  (resp.  $p \neq 11$ ) (cf. [1, Proof of Lemma 4.4]). We see  $(e_3, f_3, g_3) = (1, 30, 1)$  (resp.  $(e_{11}, f_{11}, g_{11}) = (1, 30, 1)$ ). Now we prove  $M^B(k_v) \neq \emptyset$  for a place  $v$  of  $k$  above 3 (resp. 11). Since the order of  $\mathbb{F}_{330}^\times$  (resp.  $\mathbb{F}_{1130}^\times$ ) is divisible by 4, we have  $\sqrt{-1} \in k_v$ . Then  $[1, \sqrt{-1}, 0] \in M^B(k_v)$ . Therefore  $M^B(k) \neq \emptyset$ . Since the genus of  $M^B$  is 0, we conclude  $\#M^B(k) = \infty$ .

(3) [Case  $d = 6$ ]. All the imaginary quadratic subfields of  $kW_{-6}$  are  $\mathbb{Q}(\sqrt{-6})$  and  $\mathbb{Q}(\sqrt{-31})$ , whose class numbers are 2 and 3 respectively. First assume that  $kW_{-6}$  contains the Hilbert class field  $H(-6)$  of  $\mathbb{Q}(\sqrt{-6})$ . Then  $H(-6) = \mathbb{Q}(\sqrt{-6}, \sqrt{-31})$ , because we have  $[H(-6) : \mathbb{Q}] = h_{\mathbb{Q}(\sqrt{-6})}[\mathbb{Q}(\sqrt{-6}) : \mathbb{Q}] = 4$  and  $\text{Gal}(kW_{-6}/\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/15\mathbb{Z}$ . But, in the extension  $\mathbb{Q}(\sqrt{-6}, \sqrt{-31})/\mathbb{Q}(\sqrt{-6})$ , the primes of  $\mathbb{Q}(\sqrt{-6})$  above 31 are ramified, which is a contradiction. Next assume that  $kW_{-6}$  contains the Hilbert class field  $H(-31)$  of  $\mathbb{Q}(\sqrt{-31})$ . Take a prime  $\mathcal{P}$  of  $kW_{-6}$  above 31, and let  $\mathfrak{p}_H$  (resp.  $\mathfrak{p}$ ) be the prime of  $H(-31)$  (resp.  $\mathbb{Q}(\sqrt{-31})$ ) below  $\mathcal{P}$ . Then  $e(\mathcal{P}/\mathfrak{p}_H) = 15$  because we have  $e(\mathcal{P}/\mathfrak{p}) = 15$  and  $e(\mathfrak{p}_H/\mathfrak{p}) = 1$ , where  $e(\cdot/\cdot)$  is the ramification index. Since  $[H(-31) : \mathbb{Q}(\sqrt{-31})] = h_{\mathbb{Q}(\sqrt{-31})} = 3$ , we have  $[kW_{-6} : H(-31)] = 10$ . This contradicts  $e(\mathcal{P}/\mathfrak{p}_H) = 15$ .

[Case  $d = 22$ ]. All the imaginary quadratic subfields of  $kW_{-22}$  are  $\mathbb{Q}(\sqrt{-22})$  and  $\mathbb{Q}(\sqrt{-31})$ , whose class numbers are 2 and 3 respectively. Then we are done by the same argument as in [Case  $d = 6$ ].

(4) The least  $N(\mathfrak{q}')$  for primes  $\mathfrak{q}'$  of  $kW_{-d}$ , of odd degree, whose residual characteristic  $q$  satisfies  $B \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{-q}) \not\cong M_2(\mathbb{Q}(\sqrt{-q}))$ , is  $2^5 = 32$ . Then the assertion follows from Theorem 6.2 (1) and Lemma 7.3 (1).

□

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(Keisuke Arai) Department of Mathematics, School of Engineering, Tokyo Denki University, 5 Senju Asahi-cho, Adachi-ku, Tokyo 120-8551, Japan  
*E-mail address:* `araik@mail.dendai.ac.jp`